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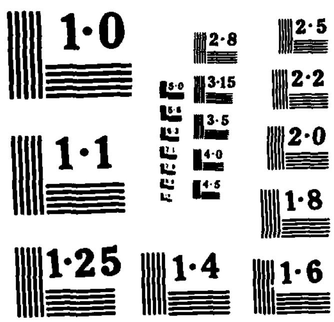

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PEAKEDNESS OF WEIGHTED AVERAGES OF JOINTLY DISTRIBUTED RANDOM VARIABLES

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by

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This note extends the Proschan (1965) result on peakedness comparison for a convex combination of i.i.d. random variables from a PF_2 density. Now the underlying random variables are jointly distributed from a Schur-concave density. The result permits a more refined discription of convergence in the Law of Large Numbers. Keywords: Cauchy distributions

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ABSTRACT

This note extends the Proschan (1965) result on peakedness comparison for a convex combination of i.i.d. random variables from a PF_2 density. Now the underlying random variables are jointly distributed from a Schur-concave density. The result permits a more refined discription of convergence in the Law of Large Numbers.

1. Introduction

Proschan (1965) shows that:

1.1 Theorem. Let f be PF_2, $f(t) = f(-t)$ for all t, $X_1, \ldots, X_n$ independently distributed with density f, $\underset{\sim}{p} \overset{m}{>} \underset{\sim}{p}'$, $\underset{\sim}{p}$, $\underset{\sim}{p}'$ not identical, $\sum_1^n p_i = 1 = \sum_1^n p_i'$. Then $\sum_1^n p_i' X_i$ is strictly more peaked than $\sum_1^n p_i X_i$.

(Definitions of majorization ($\underset{\sim}{p} \overset{m}{>} \underset{\sim}{p}'$), PF_2 density, and peakedness are presented in Section 2.) Roughly speaking, Theorem 1.1 states that a weighted average of i.i.d. random variables converges more rapidly in the case in which weights are close together as compared with the case in which the weights are diverse.

In the present note, we extend the basic univariate result to the multivariate situation in which the underlying random variables have a joint Schur-concave density. Theorem 2.3 presents the precise statement of the multivariate extension.

2. Peakedness comparisons

The theory of majorization is exploited in this section to obtain more general versions of the result of Proschan (1965). We begin with some definitions

Definition 2.1. Let $a_1 \geq \ldots \geq a_n$ and $b_1 \geq \ldots \geq b_n$ be decreasing rearrangements of the components of the vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$. We say that the vector $\underset{\sim}{b}$ is majorized by $\underset{\sim}{a}$, written $\underset{\sim}{a} \overset{m}{\geq} \underset{\sim}{b}$ if

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i$$

and $$\sum_{i=1}^{k} a_i \geq \sum_{i=1}^{k} b_i \quad \text{for } k = 1, \ldots, n-1.$$

Definition 2.2. A real valued function f defined on R^n is said to be a Schur-concave function if $f(\underset{\sim}{a}) \leq f(\underset{\sim}{b})$ whenever $\underset{\sim}{a} \overset{m}{\geq} \underset{\sim}{b}$.

A function f defined on R^n is said to be sign invariant if $f(x_1, \ldots, x_n) = f(|x_1|, \ldots, |x_n|)$. In the following Lemma, we give a peakedness comparison for random variables with a sign invariant and Schur-concave density.

Theorem 2.3. Suppose the random vector $\underset{\sim}{X} = (X_1, \ldots, X_n)$ has a Schur-concave density f. If f is sign-invariant and satisfies

$$\int_{-\infty}^{\infty} uf(u, u, x_3, \ldots, x_n)\, du < \infty \text{ for all } x_3, \ldots, x_n.$$

Then for all $t \geq 0$,

$$\Psi(a_1, \ldots, a_n) = P(\sum a_i X_i \leq t)$$

is a Schur-concave function of $\underset{\sim}{a} = (a_1, \ldots, a_n)$, $a_i \geq 0$ for all i. Equivalently, $\sum b_i X_i$ is more peaked than $\sum a_i X_i$ whenever $\underset{\sim}{a} \overset{m}{\geq} \underset{\sim}{b}$.

Proof.

Without loss of generality, we may assume that $\sum a_i = 1$. We first consider the case $n = 2$.

Let $0 \leq a \leq \frac{1}{2}$ and $\bar{a} = 1 - a$. Let $h(a) = P(\bar{a}X_1 + \bar{a}X_2 \leq t) = \int_{-\infty}^{\infty} G_{X_2|X_1=u}(\frac{t - au}{\bar{a}})\, g_1(u)\, du$ where g_1 is the marginal density of X_1 and $G_{X_2|X_1=u}$ is the conditional distribution function of X_2 given that $X_1 = u$.

Differentiation under the integral sign is permissible here, so that

$$\bar{a}^2\, h'(a) = \int_{-\infty}^{\infty} g_{X_2 | X_1 = u}\left(\frac{t - au}{\bar{a}}\right) g_1(u)\,(t-u)\; du$$

$$= \int_{-\infty}^{\infty} f\left(u, \frac{t - au}{\bar{a}}\right)(t-u)\; du.$$

$$= \int_{-\infty}^{t} f\left(u, \frac{t - au}{\bar{a}}\right)(t-u)\; du$$

$$+ \int_{t}^{\infty} f\left(u, \frac{t - au}{\bar{a}}\right)(t-u)\; du.$$

Now let $v = t - u$ in the first integral and $v = u - t$ in the second integral. We obtain

$$\bar{a}^2\, h'(a) = \int_0^{\infty} v\,\left[f\left(t - v,\; t + \frac{a}{\bar{a}}\, v\right) - f\left(t + v,\; t - \frac{a}{\bar{a}}\, v\right)\right] dv$$

$$= \int_0^{\infty} v\,\left[f\left(v - t,\; \frac{a}{\bar{a}}\, v + t\right) - f\left(v + t,\; \frac{a}{\bar{a}}\, v - t\right)\right] dv,$$

since f is sign invariant. But this is nonpositive because

$$\left(v + t,\; \frac{a}{\bar{a}}\, v - t\right) \overset{m}{\geq} \left(v - t,\; \frac{a}{\bar{a}}\, v + t\right)$$

and f is Schur-concave. Thus h(a) is increasing in a, $0 \leq a \leq \frac{1}{2}$.

The result for $n \geq 3$ now follows since

$$P\left(\sum a_i X_i \leq t\right)$$

$$= E\left[P\left(a_1 X_1 + a_2 X_2 \leq t - \sum_3^n a_i X_i \,|\, X_3, \ldots, X_n\right)\right]$$

and the conditional density $f(x_1, x_2 | x_3, \ldots, x_n)$ is also Schur-concave and sign invariant. □

Remark 2.4. To justify differentiation under the integral sign, we note that

$$\int_{-\infty}^{\infty} \left| f\left(u, \frac{t - au}{\bar{a}}\right)(t-u) \right| du$$

$$\leq \int_{-\infty}^{\infty} |t - u|\; f\left(\frac{|u - t|}{\bar{a}}, \frac{|u - t|}{\bar{a}}\right) du < \infty,$$

which follows from (2.1).

This condition is clearly not a necessary condition, but it can be easily varified for most Schur-concave multivariate distributions. For example, the multivariate Cauchy density:

$$f(x_1,\ldots, x_n) = \pi^{-(n+1)/2}\,\Gamma\,((n+1)/2)(1 + \sum_{i=1}^{n} x_i^2)^{-(n+1)/2}$$

has this property.

The following result is an immediate application of Theorem 2.3.

<u>Corollary 2.5</u>. Let $X_1,\ldots, X_n$ be random variables with joint Schur-concave density f. Let f be sign invariant and satisfy

$$\int_{-\infty}^{\infty} uf(u, u, x_2,\ldots, x_n)\, du < \infty \text{ for all } x_3,\ldots, x_n.$$

Then $\frac{1}{k}\sum_{i=1}^{k} X_i$ is increasing in peakedness as k increases from 1 to n.

<u>Proof</u>.

Let $\underset{\sim}{a}_1 = (1, 0,\ldots 0)$, $\underset{\sim}{a}_2 = (\frac{1}{2}, \frac{1}{2}, 0,\ldots 0),\ldots$ and $\underset{\sim}{a}_n = (\frac{1}{n},\ldots, \frac{1}{n})$ where each vector contains n components. Then $\underset{\sim}{a}_1 \overset{m}{\geq} \underset{\sim}{a}_2 \overset{m}{\geq} \ldots \overset{m}{\geq} \underset{\sim}{a}_n$. The result follows from Theorem 2.3. □

Suppose $\underset{\sim}{X} = (X_1,\ldots, X_n)$ and $\underset{\sim}{Y} = (Y_1,\ldots, Y_n)$ are independently distributed with respective densities f and g where both f and g are Schur-concave and sign invariant, Theorem 2.3 implies that $\sum b_i(X_i + Y_i)$ is more peaked than $\sum a_i(X_i + Y_i)$ whenever $\underset{\sim}{a} \overset{m}{\geq} \underset{\sim}{b}$. This is true because the convolution of Schur-concave functions is Schur-concave. However, if $Y_1,\ldots, Y_n$ are i.i.d. Cauchy, then the joint density given by

$$g(X_1,\ldots, X_n) = (\frac{a}{\pi})^n \prod_{i=1}^{n} (1 + a^2X_i^2)^{-1},\ a > 0,$$

is not Schur-concave. Theorem 2.7 below, we give conditions on f for which (2.2) holds. First we prove the following Lemma.

Lemma 2.6. Let $\underset{\sim}{X} = (X_1, \ldots, X_n)$ and $\underset{\sim}{Y} = (Y_1, \ldots, Y_n)$ be independently distributed with respective densities f_1 and f_2. Suppose $f_i(t_1, \ldots, t_n)$ is symmetric with respect to zero and nonincreasing in each argument for $t_k > 0$, $k = 1, \ldots, n$, and for $i = 1,2$.

Let $\sum_1^n b_i X_i$ be more peaked than $\sum_1^n a_i X_i$ and

$\sum_1^n b_i Y_i$ be more peaked than $\sum_1^n a_i Y_i$ where $a_i \geq 0$ and $b_i \geq 0$ for $i = 1, \ldots, n$. Then

$$\sum_1^n b_i(X_i + Y_i) \text{ is more peaked than is } \sum_1^n a_i(X_i + Y_i).$$

Proof.

This result follows immediately from the Lemma of Birnbaum (1948) by noting that the random variables $\sum_1^n a_i X_i$, $\sum_1^n a_i Y_i$, $\sum_1^n b_i X_i$, and $\sum_1^n b_i Y_i$ have symmetric and unimodal densities. □

The following theorem identifies a different class of densities for which the conclusion of Theorem 2.3 holds.

Theorem 2.7. Suppose that the random vector $\underset{\sim}{X} = (X_1, \ldots, X_n)$ has a Schur-concave sign-invariant density f. Let f be nonincreasing in each argument over the positive values and satisfy (2.1). Let $Y_1, \ldots, Y_n$ be i.i.d. Cauchy with joint density g as given in (2.3). Let $\underset{\sim}{X}$ and $\underset{\sim}{Y} = (Y_1, \ldots, Y_n)$ be independent, and $\underset{\sim}{a} \overset{m}{\geq} \underset{\sim}{b}_i$ where $a_i \geq 0$, $b_i \geq 0$ for all i and $1 = \sum_1^n a_i = \sum_1^n b_i$. Then

$$\sum_1^n b_i(X_i + Y_i) \text{ is more peaked than is } \sum_1^n a_i(X_i + Y_i).$$

Proof.

We use the fact that $\sum_1^n a_i Y_i$, $\sum_1^n b_i Y_i$ have the same distribution as does Y_1. From Theorem 2.3, $\sum_1^n b_i X_i$ is more peaked than is $\sum_1^n a_i X_i$. The result now follows from Lemma 2.6. □

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[2] Proschan, F. (1965). Peakedness of distributions of convex combinations. Ann. Math. Statist. 36. 1703-1706.

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